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AN ICER APPROACH TO THE THEORY OF MINIMAL FLOWS

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ABSTRACT

We study minimal flows by studying a universal minimal flow, invariant closed equivalence relations (icers) on it, subgroups of its group of automorphisms, and the interplay among these objects. As examples of this approach we discuss and give short proofs of some standard results on distal flows. We end with a statement of the Furstenberg structure theorem from this point of view.

1. Introduction

In this paper we expound the classical theory of minimal flows from the point of view of invariant closed equivalence relations (icers) on the universal minimal set M. By a **flow** we mean a compact Hausdorff space, X, upon which a topological group T (fixed for the duration of the paper) acts on the right. In the situations considered here there is no loss of generality if T is given the discrete topology. As usual a flow is **point transitive** if the orbit closure of some point in X is all of X, and **minimal** if the orbit closure of every point in X is all of X.

The Stone–Cech compactification, βT, of T possesses a natural semigroup structure which makes βT into a point transitive flow. This flow is universal in the sense that any point transitive flow is a homomorphic image of βT.

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Moreover, any minimal right ideal $M \subset \beta T$ is universal for minimal flows. Thus every minimal flow X, being a homomorphic image of M, may be written in the form $X = M/R$ for some icer R on M.

The approach taken here is to understand the properties of M/R by studying the icer R. One of the key tools used is the group, G, of automorphisms of M. Icers are constructed and analyzed using subgroups of G. In particular the group, $G(R)$, associated to an icer is defined as a subgroup of G. In [E1] G was viewed instead as a subset Mu of M, where $u \in M$ was a fixed idempotent. The group of a minimal flow was then defined up to conjugacy as a subgroup of Mu. The point of view taken here eliminates the asymmetrical treatment of the idempotents in M. However, this new approach cannot be a mere translation of the results in [A1], [E1] and [G1], since the same flow, X, may have several different representations in the form M/R; that is, we could have $M/S \cong X \cong M/R$ with $R \neq S$.

This approach gives rise to new concepts and problems. We discuss some of these. The reader is assumed to be acquainted with the material in [A1], [E1] and [G1].

2. Notation and definitions

We will be considering minimal flows (X, T) for a fixed group T.

2.1 Notation: Let βT denote the Stone–Cech compactification of the discrete group T. We fix a minimal right ideal $M \subset \beta T$. We will use

$$J = \{u \in M \mid u^2 = u\}$$

to denote the set of idempotents in M. We will use

$$G = \{\alpha \colon M \to M \mid \alpha \text{ is an automorphism}\}$$

to denote the set of automorphisms of M.

2.2 Definition and Notation: Let $R \subset M \times M$. The actions of T, βT, and M on M extend in the obvious way to diagonal actions on $M \times M$. When R is closed and invariant under T (so that $(p,q)t = (pt, qt) \in R$ for all $(p,q) \in R$), R is also invariant under the actions of βT and M (so that $Rz = R$ for all $z \in \beta T$). We refer to a closed invariant equivalence relation as an **icer**. Note that in this case M/R is compact, and the action of T on M induces an action of T on M/R, so that M/R is a flow.

Indeed, any minimal flow X is a homomorphic image of M, so we can write $X \cong M/R$ for some T-invariant closed equivalence relation (icer) R, on M. We will study minimal flows by studying the corresponding icers on M. It will be convenient to use the following notation.

Let R, S be icers on M with $R \subset S$. Then

$$\pi_R \colon M \to M/R \quad \text{and} \quad \pi_S^R \colon M/R \to M/S$$

denote the canonical identification maps.

We will make repeated use of the following proposition (whose proof appears in slightly different notation in [E1]).

2.3 Proposition: *Let $M \subset \beta T$ be a fixed minimal ideal. Then every element $p \in M$ can be written uniquely in the form $p = \alpha(u)$ for some automorphism $\alpha \in G$ and idempotent $u \in J$. Moreover, the semigroup structure on M is given by:*

$$\alpha(u)\beta(v) = \alpha(\beta(v)).$$

Given an automorphism $\alpha \in G$, and an icer R on M, it is natural to consider the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & M \\ {\scriptstyle \pi_R}\downarrow & & \downarrow{\scriptstyle \pi_R} \\ M/R & \xrightarrow{?} & M/R \end{array}$$

and ask: When can we fill in the question mark so as to give an automorphism of M/R? In other words, when does α "descend" to an automorphism of M/R? This will happen when

$$(\alpha(p), \alpha(q)) \in R \iff (p, q) \in R.$$

Writing $\alpha(p,q) = (\alpha(p), \alpha(q))$, this condition amounts to saying that $\alpha(R) = R$. Similarly, α descends to the identity on M/R when $(p, \alpha(p)) \in R$ for all $p \in M$. These facts motivate the following definitions:

2.4 Definition: Let R be an icer on M. We denote

$$aut(R) = \{\alpha \in G \mid \alpha(R) = R\}.$$

We define the **group of R** by

$$G(R) = \{\alpha \in G \mid gr(\alpha) \subset R\},$$

where the graph of $\alpha \in G$ is given by

$$gr(\alpha) = \{(p, \alpha(p)) \mid p \in M\} \subset M \times M.$$

It is straightforward to check, for instance by using the remarks above, that $G(R)$ is a normal subgroup of $aut(R)$.

Having defined the group of an icer R as a subgroup of G, it is natural to ask: For which subgroups $A \subset G$ does there exist an icer R on M such that $G(R) = A$? The answer is provided by the so-called τ-topology on G. Namely, such an R exists if and only if the subgroup A is τ-closed. The construction of this topology and the proof of this result from the point of view taken here is interesting but not the subject of this paper. Here we will just make use of the result together with the fact that the product of τ-closed subsets of G is again τ-closed. We state this result for reference.

2.5 The τ-topolology on G: There exists a compact T_1 topology on G such that:

(a) A is a closed subgroup of G if and only if

$$A = G(R) \quad \text{where } R = \overline{gr(A)} \equiv \overline{\cup\{gr(\alpha) \mid \alpha \in A\}};$$

(b) if A, B are closed in G, then AB is closed.

3. Some basic results

One of our main themes is that icers on M (and hence flows) can be studied using the group G and its subgroups. We begin with a basic result which facilitates this approach.

3.1 PROPOSITION: *Let R be an icer on M. Then*

$$\begin{aligned} R &= \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(R), \text{ and } \alpha(u,v) \in R\} \\ &= \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(R), \text{ and } \beta(u,v) \in R\}. \end{aligned}$$

Proof: Let $(\alpha(u), \beta(v)) \in R$. Then

$$(\alpha\beta^{-1}(v), v) = (\alpha(u), \beta(v))\beta^{-1}(v) \in R\beta^{-1}(v) \subset R.$$

Hence $gr(\alpha\beta^{-1}) \subset R$ and $\alpha\beta^{-1} \in G(R)$. Moreover,

$$(\alpha(v), \beta(v)) = (\alpha(u), \beta(v))v \in Rv \subset R,$$

and since R is transitive $(\alpha(u), \alpha(v)) \in R$.

On the other hand, assume that $(\alpha(u), \alpha(v)) \in R$ and $\alpha\beta^{-1} \in G(R)$. Then

$$(\beta(v), \alpha(v)) = (\beta(v), \alpha\beta^{-1}(\beta(v))) \in gr(\alpha\beta^{-1}) \subset R.$$

Since R is transitive it follows that $(\alpha(u), \beta(v)) \in R$. We have shown that

$$R = \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(R), \text{ and } \alpha(u,v) \in R\};$$

the other equality follows immediately from the fact that R is symmetric.

3.2 PROPOSITION: *Let R, S be icers on M. Then:*

(a) *if φ: $M/R \to M/S$ is a homomorphism, then there exists $\alpha \in G$ with $\alpha(R) \subset S$ and $\varphi \circ \pi_R = \pi_S \circ \alpha$;*

(b) *if $\alpha \in G$ with $\alpha(R) \subset S$, then there exists a homomorphism φ: $M/R \to M/S$ such that $\varphi \circ \pi_R = \pi_S \circ \alpha$.*

Proof: (a) Let φ: $M/R \to M/S$ be a flow homomorphism. Then $gr(\varphi)$ is a minimal subset of $M/R \times M/S$, so there exists a minimal set $Y \subset M \times M$ with $(\pi_R \times \pi_S)(Y) = gr(\varphi)$. But the only minimal subsets of $M \times M$ are graphs of automorphisms, so there exists $\alpha \in G$ with $Y = gr(\alpha)$. Now it is immediate that $\varphi \circ \pi_R = \pi_S \circ \alpha$ and $\alpha(R) \subset S$.

(b) Let $\alpha \in G$ with $\alpha(R) \subset S$, and $(p,q) \in R$. Then $(\alpha(p), \alpha(q)) \in S$ and hence $\pi_S(\alpha(p)) = \pi_S(\alpha(q))$. It follows that

$$\varphi: M/R \to M/S$$
$$pR \to \alpha(p)S$$

is a well-defined homomorphism with $\varphi \circ \pi_R = \pi_S \circ \alpha$.

3.3 COROLLARY: *Let R, S be icers on M. Then $M/R \cong M/S$ if and only if $\alpha(R) = S$ for some $\alpha \in G$.*

3.4 COROLLARY: *Let R, S be icers on M such that $M/R \cong M/S$. Assume that $\alpha(R) = R$ for all $\alpha \in G$. Then $R = S$. Thus in this case the representation of X in the form M/R is unique.*

3.5 Definition: We say that an icer R on M is **regular** if $\alpha(R) = R$ for all $\alpha \in G$. Thus R is regular if and only if $aut(R) = G$.

3.6 Remarks: In [A2] Auslander defined the minimal flow X to be regular if, given $x, y \in X$, there exists a homomorphism φ: $X \to X$ such that $\varphi(x)$ and

y are proximal. In fact, R is regular in the sense of 3.5 if and only if M/R is regular in Auslander's sense.

Proof: Suppose that R is regular in the sense of 3.5, and let $x = \pi_R(\alpha(u))$, $y = \pi_R(\beta(v)) \in M/R$. Then $\beta\alpha^{-1} \in G$ induces an automorphism

$$\begin{aligned}\varphi\colon M/R &\to M/R\\ pR &\to \beta\alpha^{-1}(p)R.\end{aligned}$$

Clearly $\varphi(x)v = yv$ and hence $\varphi(x)$ is proximal to y.

On the other hand, suppose that M/R is regular in Auslander's sense, and let $\alpha \in G$ and $u \in J$. Then there exists a homomorphism φ: $M/R \to M/R$ such that $\varphi(\pi_R(u))$ is proximal to $\pi_R(\alpha(u))$. Thus for some $b \in \beta T$,

$$\varphi(\pi_R(ub)) = \varphi(\pi_R(u))b = \pi_R(\alpha(u))b = \pi_R(\alpha(ub)),$$

and hence $\varphi \circ \pi_R = \pi_R \circ \alpha$. It then follows immediately that $\alpha(R) \subset R$.

3.7 PROPOSITION: *Let f: $X \to Y$ be a homomorphism of minimal flows. Assume that g: $X \cong M/R$ where R is an icer on M. Then there exists an icer S on M and an isomorphism h: $Y \cong M/S$ such that*

(a) $R \subset S$,

(b) $\pi_S^R \circ g = h \circ f$; *that is, the folowing diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{g} & M/R \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_S^R} \\ Y & \xrightarrow{h} & M/S \end{array}$$

This shows that in studying extensions $X \to Y$ we may assume that we are looking at the canonical map π_S^R: $M/R \to M/S$ where $R \subset S$ are icers on M. Thus we say that M/R is an **extension** of M/S if $R \subset S$.

Proof: Let $S = \{(p,q) \in M \times M \mid f(g^{-1}(\pi_R(p))) = f(g^{-1}(\pi_R(q)))\}$. Then S is an icer on M with $R \subset S$. The map $f \circ g^{-1} \circ \pi_R$: $M \to Y$ induces an isomorphism σ: $M/S \to Y$. Setting $h = \sigma^{-1}$ gives the desired result.

4. The relative product

4.1 Definition: Let R, S be any relations on M. We define the **relative product $R \circ S$, of R and S** by

$$R \circ S = \{(p,q) \in M \times M \mid \text{there exists } r \in M \text{ with } (p,r) \in R \text{ and } (r,q) \in S\}.$$

4.2 Remark: Note that in general the relative product $R \circ S$ of two closed equivalence relations is closed and reflexive, but need not be symmetric or transitive. It is clear that when $R \circ S$ is an icer, it is the smallest icer containing both R and S. Thus in this case $M/(R \circ S)$ is the infimum of the flows M/R and M/S. In this section we will be interested in conditions on R and S which guarantee that $R \circ S$ is an icer.

4.3 PROPOSITION: *Let R, S be icers on M such that $R \circ S$ is an icer on M. Then $G(R \circ S) = G(R)G(S)$.*

Proof: Since $R, S \subset R \circ S$, both $G(R)$ and $G(S)$ are contained in $G(R \circ S)$. But the latter is a group, so $G(R)G(S) \subset G(R \circ S)$.

In order to show that $G(R \circ S) \subset G(R)G(S)$, let $\alpha \in G(R \circ S)$ and $u^2 = u \in M$. Then $(u, \alpha(u)) \in R \circ S = S \circ R$, so there exists $\beta(v) \in M$ with $(u, \beta(v)) \in S$ and $(\beta(v), \alpha(u)) \in R$. Hence $\beta \in G(S)$ and $\beta\alpha^{-1} \in G(R)$. Thus $\alpha = (\alpha\beta^{-1})\beta = (\beta\alpha^{-1})^{-1}\beta \in G(R)G(S)$.

4.4 Notation: It will be convenient to use the notation

$$P_0 = \{(\alpha(u), \alpha(v)) \mid \alpha \in G, (u, v) \in J \times J\} = \bigcup_{\alpha \in G} \alpha(J \times J).$$

Note that P_0 is an equivalence relation.

4.5 PROPOSITION: *Let R, S be icers on M such that:*
(1) $P_0 \cap S \subset P_0 \cap R$,
(2) $G(S) \subset G(R)$.
Then $S \subset R$.

Proof: Let $(\alpha(u), \beta(v)) \in S$. Then by Proposition 3.1, $\alpha\beta^{-1} \in G(S)$ and $\alpha(u, v) \in S$. Our assumptions now imply that $\alpha\beta^{-1} \in G(R)$ and $\alpha(u, v) \in P_0 \cap S \subset P_0 \cap R$. Thus, again by 3.1, $(\alpha(u), \beta(v)) \in R$.

4.6 PROPOSITION: *Let*
(1) *R, S be icers on M,*
(2) *$G(S) \subset aut(R)$,*
(3) *$S \cap P_0 \subset R \cap P_0$.*
Then $R \circ S$ is an icer on M.

Proof: As we remarked in 4.2, $R \circ S$ is always closed invariant and reflexive when R and S are icers, so we need only show that it is symmetric and transitive. Let

$(\alpha(u),\gamma(w)) \in R\circ S$. Then there exists $\beta \in G$ and $v \in J$ with $(\alpha(u),\beta(v)) \in R$ and $(\beta(v),\gamma(w)) \in S$. Hence by 3.1,

$$\beta\alpha^{-1} \in G(R), \quad (\alpha(u),\alpha(v)) \in R, \quad \gamma\beta^{-1} \in G(S) \quad \text{and} \quad (\gamma(v),\gamma(w)) \in S\cap P_0.$$

Thus

$$\gamma\alpha^{-1} = \gamma\beta^{-1}\beta\alpha^{-1} \in G(S)G(R).$$

Assumption (2) together with the fact that $G(R)$ is normal in $aut(R)$ implies that $G(R)G(S) = G(S)G(R)$. Therefore, there exist $\rho \in G(R)$ and $\nu \in G(S)$ with $\gamma\alpha^{-1} = \rho\nu$. Since $\nu \in G(S) \subset aut(R)$, it follows that

$$(\alpha(u),\nu(\alpha(u))) \in S \quad \text{and} \quad (\nu(\alpha(u)),\nu(\alpha(v))) = \nu(\alpha(u),\alpha(v)) \in \nu(R) = R.$$

Combining these gives that $(\alpha(u),\nu(\alpha(v))) \in S\circ R$. On the other hand, $\rho \in G(R)$, so

$$(\nu(\alpha(v)),\gamma(v)) = (\nu(\alpha(v)),\rho\nu(\alpha(v))) \in R.$$

Combining the last two statements gives $(\alpha(u),\gamma(v)) \in S\circ R\circ R = S\circ R$. Finally,

$$(\gamma(v),\gamma(w)) \in S\cap P_0 \subset R\cap P_0,$$

so $(\alpha(u),\gamma(w)) \in S\circ R\circ R = S\circ R$. We have now shown that $R\circ S \subset S\circ R$. But then

$$S\circ R = (R\circ S)^{-1} \subset (S\circ R)^{-1} = R\circ S,$$

and hence $R\circ S = S\circ R$ is symmetric. It follows immediately that $R\circ S$ is transitive because

$$(R\circ S)\circ(R\circ S) = R\circ S\circ S\circ R = R\circ S\circ R = R\circ R\circ S = R\circ S.$$

4.7 PROPOSITION: *Let*
(1) R, S *be icers on* M,
(2) $G(R)G(S) = G(S)G(R)$,
(3) $R\cap P_0 = S\cap P_0$.
Then $R\circ S$ *is an icer on* M.

Proof: Let $(\alpha(u),\gamma(w)) \in R\circ S$. Then there exists $\beta \in G$ and $v \in J$ with $(\alpha(u),\beta(v)) \in R$ and $(\beta(v),\gamma(w)) \in S$. Hence $\beta\alpha^{-1} \in G(R)$ and $\gamma\beta^{-1} \in G(S)$, from which it follows that

$$\gamma\alpha^{-1} = \gamma\beta^{-1}\beta\alpha^{-1} \in G(S)G(R) = G(R)G(S).$$

Thus there exist $\rho \in G(R)$ and $\nu \in G(S)$ with $\gamma\alpha^{-1} = \rho\nu$. Now $(\alpha(u), \alpha(v)) \in R \cap P_0 = S \cap P_0$, so

$$(\alpha(u), \nu(\alpha(v))) \in S$$

because $\nu \in G(S)$. Also, $(\nu(\alpha(v)), \gamma(v)) = (\nu(\alpha(v)), \rho\nu(\alpha(v))) \in R$ because $\rho \in G(R)$. Finally,

$$(\gamma(v), \gamma(w)) \in S \cap P_0 = R \cap P_0,$$

so $(\alpha(u), \gamma(w)) \in S \circ R \circ R = S \circ R$. This shows that $R \circ S \subset S \circ R$. It follows as in the proof of 4.6 that $R \circ S$ is an icer.

5. Distal flows

5.1 Definition: Let R be an icer on M. We say that R is **distal** if $X = M/R$ is a distal flow.

5.2 Remark: Let $(\alpha(u), \alpha(v)) \in P_0$. Then for any $p \in M$,

$$\pi_R(\alpha(u))p = \pi_R(\alpha(u)p) = \pi_R(\alpha(p)) = \pi_R(\alpha(v)p) = \pi_R(\alpha(v))p,$$

so $\pi_R(\alpha(u))$ and $\pi_R(\alpha(v))$ are proximal points in M/R. Thus when R is distal, $\pi_R(\alpha(u)) = \pi_R(\alpha(v))$ and hence $(\alpha(u), \alpha(v)) \in R$. This shows that when R is distal, $P_0 \subset R$. In fact, as we state for emphasis in the following proposition, this property gives a characterization of distal icers.

5.3 PROPOSITION: *Let R be an icer on M. Then $X = M/R$ is a distal flow if and only if $P_0 \subset R$.*

Proof: 5.2 shows that if R is distal, then $P_0 \subset R$. On the other hand, suppose that $P_0 \subset R$ and let $\alpha(u), \beta(v) \in M$ with $\pi_R(\alpha(u))$ proximal to $\pi_R(\beta(v))$. Then

$$\pi_R(\alpha(ub)) = \pi_R(\alpha(u))b = \pi_R(\beta(v))b = \pi_R(\beta(vb)),$$

for some $b \in \beta T$. But $(\alpha(u), \alpha(v)) \in P_0 \subset R$, so

$$(\alpha(ub), \alpha(vb)) = (\alpha(u), \alpha(v))b \subset Rb \subset R.$$

Since R is transitive it follows that $(\alpha(vb), \beta(vb)) \in R$, and hence $(\alpha(u), \beta(u)) \in R$ (because $u \in M = vbM$). Using the fact that $(\beta(u), \beta(v)) \in P_0 \subset R$, it now follows that $(\alpha(u), \beta(v)) \in R$ and $\pi_R(\alpha(u)) = \pi_R(\beta(v))$.

5.4 Definition: Let

$$\mathcal{D} = \{R \mid R \text{ is a distal icer on } M\}$$

and set $S_d = \bigcap_{R\in\mathcal{D}} R$. It follows immediately from 5.3 that S_d is a distal icer. By construction, R is a distal icer if and only if $S_d \subset R$.

5.5 PROPOSITION: *Let $D = G(S_d)$. Then:*

(a) *S_d is regular,*
(b) *D is a normal subgroup of G,*
(c) *$S_d = \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in D, \text{ and } u, v \in J\}$.*

Proof: (a) If R is distal, then for any $\alpha \in G$, $P_0 = \alpha(P_0) \subset \alpha(R)$, and hence $\alpha(R)$ is distal. Thus

$$\alpha(S_d) = \alpha\Big(\bigcap_{R\in\mathcal{D}} R\Big) = \bigcap_{R\in\mathcal{D}} \alpha(R) = S_d.$$

(b) $G(S_d)$ is normal in $aut(S_d)$, which is G by part (a).
(c) By 3.1, $S_d = \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in D, \text{ and } (u,v) \in \alpha^{-1}(S_d)\}$. Thus the result follows from the fact that $\alpha(u,v) \in P_0 \subset S_d$ for all $\alpha \in G$ and $(u,v) \in J \times J$.

5.6 PROPOSITION: *Let $X = M/R$. Then:*

(a) *$R \circ S_d$ is an icer on M,*
(b) *$G(R \circ S_d) = G(R)D$.*

Proof: (a) Since S_d is regular, $G(R) \subset G = aut(S_d)$. Since S_d is distal, $R \cap P_0 \subset P_0 = S_d \cap P_0$. It now follows from 4.6 that $R \circ S_d$ is an icer.
(b) This follows from part (a) and 4.3.

5.7 PROPOSITION: *Let*

(1) *R, S be icers on M,*
(2) *S be distal,*
(3) *$G(R)G(S)$ be a group.*

Then

(a) *$R \circ S$ is an icer on M,*
(b) *$G(R \circ S) = G(R)G(S)$.*

Proof: (a) $\overline{gr(G(R)G(S))}$ is an icer on M by (3) and 2.5. Thus

$$N = \overline{gr(G(R)G(S))} \circ S_d$$

is an icer on M (by 5.6). We will show that $N = R \circ S$ and hence that $R \circ S$ is an icer. First note that

$$gr(G(R)G(S)) = gr(G(S)G(R)) \subset R \circ S,$$

so that $N \subset R \circ S \circ S_d = R \circ S$ (because $S_d \subset S$ by (2)). On the other hand,

$$\begin{aligned} N &= \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(N), \alpha(u,v) \in N\} \\ &= \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(N)\} \end{aligned}$$

since $P_0 \subset S_d \subset N$. Therefore

$$\begin{aligned} R &= \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(R), \alpha(u,v) \in R\} \\ &\subset \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(N)\} = N \end{aligned}$$

and similarly $S \subset N$. Since N is an equivalence relation it follows that $R \circ S \subset N$ and hence $R \circ S = N$.

(b) This follows from part (a) and 4.3.

5.8 Proposition: *Let*

(1) $\mathcal{D} = \{R \mid R$ *is a distal icer on* $M\}$,

(2) $\mathcal{G} = \{H \mid H$ *is a closed subgroup of* G *with* $D \subset H\}$.

(3)
$$\begin{aligned} \varphi\colon \mathcal{D} &\to \mathcal{G} \\ N &\to G(N), \end{aligned}$$

(4)
$$\begin{aligned} \psi\colon \mathcal{G} &\to \mathcal{D} \\ A &\to S_d \circ \overline{gr(A)}. \end{aligned}$$

Then

(a) φ *is bijective, its inverse being the map* ψ,

(b) $\psi(A)$ *is regular if and only if* A *is normal.*

Proof: The fact that φ is injective follows immediately from 4.5. The rest of the proof is left to the reader.

6. Distal extensions

6.1 Definition: Let R, S be icers on M with $R \subset S$. We say that $R \subset S$ is **distal** if $\pi_R\colon M/R \to M/S$ is distal. In other words, $R \subset S$ is a distal extension if M/R is a distal extension of M/S.

In analogy with Proposition 5.3, the following proposition gives a characterization of distal extensions. We leave the proof to the reader.

6.2 PROPOSITION: *Let $R \subset S$ be icers on M. Then $R \subset S$ is a distal extension if and only if $R \cap P_0 = S \cap P_0$.*

6.3 COROLLARY: *Let R, N, S be icers on M with $R \subset N \subset S$. Then $R \subset S$ is distal if and only if $R \subset N$ and $N \subset S$ are both distal.*

Proof: This follows immediately from 6.2.

6.4 PROPOSITION: *Let $\mathcal{N}$ be a collection of distal extensions of R. Then $S = \bigcap \mathcal{N}$ is a distal extension of R.*

Proof: By 6.2, $N \cap P_0 = R \cap P_0$ for all $N \in \mathcal{N}$. Thus $S \cap P_0 = \bigcap_{N \in \mathcal{N}}(N \cap P_0)$ $= \bigcap_{N \in \mathcal{N}}(R \cap P_0) = R \cap P_0$, and S is a distal extension of R.

6.5 Definition: Let R be an icer on M, and $\mathcal{R}$ the collection of distal extensions of R. Then we define $R^* = \bigcap \mathcal{R}$. Note that by 6.4, R^* is a distal extension of R.

6.6 PROPOSITION: *Let N, R be icers on M with $N \subset R$. Then N is a distal extension of R if and only if $R^* \subset N$.*

Proof: $\Rightarrow$ This is clear from the definition of R^*.

$\Leftarrow$ If $R^* \subset N \subset R$, then since $R^* \subset R$ is distal, it follows from 6.3 that $R^* \subset N$ and $N \subset R$ are both distal.

6.7 Remark: Let R be an icer on M. Then it is easy to check that $(\alpha(R))^* = \alpha(R^*)$ for all $\alpha \in G$. It follows immediately from this that $aut(R) \subset aut(R^*)$.

6.8 PROPOSITION: *Let R, N, S be icers on M such that:*

(1) *$S, N \subset R$,*

(2) *N is a distal extension of R,*

(3) *$G(S) \subset G(N)$.*

Then $S \subset N$.

Proof: Using assumption (2) and 6.2, we see that $P_0 \cap S \subset P_0 \cap R = P_0 \cap N$. Thus

$$S = \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(S), \alpha(u, v) \in S\}$$
$$\subset \{(\alpha(u), \beta(v)) \mid \alpha\beta^{-1} \in G(N), \alpha(u, v) \in N\} = N.$$

6.9 PROPOSITION: *Let*
(1) *R, N be icers on M,*
(2) *$N \subset R$.*
Then $R^* \circ N$ is an icer on M.

Proof: $G(N) \subset G(R) \subset aut(R) \subset aut(R^*)$ by 2.4 and 6.7. Moreover,

$$N \cap P_0 \subset R \cap P_0 = R^* \cap P_0.$$

Thus it follows from 4.6 that $R^* \circ N$ is an icer.

6.10 PROPOSITION: *Let*
(1) *N, R, S be icers on M,*
(2) *$S \subset R$ be distal,*
(3) *$G(N)G(S)$ be a group,*
(4) *$N \subset R$.*
Then $N \circ S$ is an icer on M.

Proof: $\overline{gr(G(N)G(S))}$ is an icer on M by (3) and 2.5. In addition,

$$\overline{gr(G(N)G(S))} \subset \overline{gr(G(R))} \subset R,$$

so $N_0 = \overline{gr(G(N)G(S))} \circ R^*$ is an icer on M (by 6.9). We will show that $N_0 = N \circ S$ and hence that $N \circ S$ is an icer. First note that

$$gr(G(N)G(S)) \subset N \circ S,$$

so that $N_0 \subset N \circ S \circ R^* = N \circ S$ (because $R^* \subset S$ by (2) and 6.6). On the other hand, $R^* \subset N_0 \subset R$, so N_0 is a distal extension of R by 6.6. But $G(N) \subset G(N_0)$ and $G(S) \subset G(N_0)$, so it follows from 6.8 that $N \subset N_0$ and $S \subset N_0$. Since N_0 is an equivalence relation, it follows that $N \circ S \subset N_0$ and hence $N \circ S = N_0$.

6.11 PROPOSITION: *Let*
(1) *S be a distal extension of R,*
(2) *$\mathcal{N} = \{N \mid N$ is an icer with $S \subset N \subset R\}$,*
(3) *$\mathcal{H} = \{H \mid H$ is a closed subgroup of G with $G(S) \subset H \subset G(R)\}$.*
Then

$$\varphi\colon \mathcal{N} \to \mathcal{H}$$
$$N \to G(N)$$

is bijective, its inverse being the map ψ *defined by* $\psi(H) = S \circ \overline{gr(H)}$ *for all* $H \in \mathcal{H}$.

Proof: For any $H \in \mathcal{H}$, $\overline{gr(H)} \subset \overline{gr(G(R))} \subset R$. Since $G(S) \subset H = G(\overline{gr(H)})$, $G(S)H = H$ is a group and it follows from 6.10 that $S \circ \overline{gr(H)}$ is an icer. Thus the map ψ is well-defined. Indeed $\varphi(\psi(H)) = G(S \circ \overline{gr(H)}) = G(S)H = H$.

On the other hand, if $N \in \mathcal{N}$, then $\psi(\varphi(N)) = S \circ \overline{gr(G(N))}$. But $S \circ \overline{gr(G(N))}$ is a distal extension of R whose group is $G(N)$, so it follows by applying 6.8 twice that $S \circ \overline{gr(G(N))} = N$.

7. The Furstenberg structure theorem

Finally, we describe the Furstenberg structure theorem in this context; here the so-called Furstenberg towers are constructed in a natural way using relative products.

7.1 Definition: Let $X = M/N$ and $Y = M/R$ be flows with $N \subset R$ icers. We say that X is an **almost periodic extension of** Y ($N \subset R$ is **almost periodic**) if: X is a distal extension of Y and $G(R)' \subset G(N)$.

7.2 THEOREM (Furstenberg Structure Theorem): *Let*

(1) $X = M/R$ *be a minimal distal flow,*

(2) ν *be the smallest ordinal with* $G^\nu \subset G(R)$,

(3) $G^0 = G$, $R_\alpha = \overline{gr(G^\alpha)} \circ R$ *for all* $\alpha \leq \nu$.

Then

(a) R_α *is an icer for all* α,

(b) $R_0 = M \times M$,

(c) $R_\nu = R$,

(d) $R_{\alpha+1} \subset R_\alpha$ *is an almost periodic extension for all* $\alpha + 1 \leq \nu$,

(e) $R_\alpha = \bigcap_{\beta<\alpha} R_\beta$ *for all limit ordinals* $\alpha \leq \nu$.

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